SINGULAR OSCILLATORY INTEGRAL OPERATORS

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Outline of the Talk

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Reminiscences about the 70's

Pseudo-differential operators

- Calculus of Kohn-Nirenberg, inspired by Kohn's L^2 solution of the $\bar{\partial}$ and $\bar{\partial}_b$ problems; subelliptic estimates, weakly pseudoconvex domains
- Exotic classes $S^m_{\rho,\delta}$ of Hörmander and $S^{M,m}_{\Phi,\phi}$ of Beals-Fefferman
- The almost-orthogonality lemma of Cotlar-Stein, L^2 boundedness theorem of Calderón-Vaillancourt

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Fourier integral operators

- Early ideas of Maslov and Egorov
- Theory of Hörmander and Duistermaat-Hörmander for real phases
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New singular integral operators

- Folland-Stein's fundamental solution for $\bar{\partial}_b$
- Greiner-Stein's L^p estimates for the $\bar{\partial}$ Neumann problem
- Rothschild-Stein's fundamental solution for $\sum_{j=1}^N X_j^2 + iX_0$
- Fefferman's expansion for the Bergman kernel, subsequently simplified by Kerzman-Stein, and refined by Boutet de Monvel-Sjöstrand.

Green's function for the $\bar\partial$-Neumann problem

The model case is the Siegel upper half-space $U = \{(z, z_{n+1} \in \mathbf{C}^{n+1};\ \mathrm{Im}\, z_{n+1} > |z|^2\}$, which can be identified with $H_n \times \mathbf{R}_+$ via $(z, z_{n+1}) \leftrightarrow (\zeta, \rho)$, $\zeta = (z, t)$, $t = \mathrm{Re}\, z_{n+1}$, $\rho = \mathrm{Im}\, z_{n+1} - |z|^2$. Here H_n is the Heisenberg group

$$H_n = \{\mathbf{C}^n \times \mathbf{R}; (z, t) \cdot (z', t') = (z + z', t + t' + 2\mathrm{Im}\, z\bar z')\}.$$

The $\bar\partial$-Neumann problem is the following boundary value problem

$$\Box u = f \quad \text{on} \quad H_n \times \mathbf{R}_+, \quad (\partial_\rho + i\partial_t)u = 0 \quad \text{when } \rho = 0.$$

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An explicit formula for the Green's function

$$u(\zeta, \rho) = \int_{H_n \times \mathbf{R}_+} N(\zeta^{-1} \cdot \eta, |\rho - \mu|) f(\eta, \mu) - \int_{H_n \times \mathbf{R}_+} K(\zeta^{-1} \cdot \eta, \rho + \mu) f(\eta, \mu)$$

where

$$N(\zeta, \rho) \sim \frac{1}{(2|z|^2 + t^2 + \rho^2)^n}, \qquad K(\zeta, \rho) \sim \frac{1}{(2|z|^2 + t^2 + \rho^2)^k (2|z|^2 + \rho - it)^\ell}$$

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Key features of K

- $K(\zeta,\rho)$ is a mixture of elliptic and parabolic homogeneities
- $K \in C^\infty(U \setminus 0)$, but K has hidden singularities along $t = \rho = 0$.

A distribution of hypersurfaces

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Propagation of hidden singularities along Ω_ζ

$$D^2 u_+(\zeta,\rho) = \int_0^\infty \int_{-\infty}^\infty \left\{ \int_{\Omega_\zeta} K_{\zeta,\eta}(z-w,\rho+\mu) T_s f(\eta,\mu)\, d\sigma_{\Omega_\zeta}(\eta) \right\} ds d\mu,$$

where $\zeta = (z,t)$, $\eta = (w,s)$, and $K_{\zeta,\eta}(w,\rho)$ is a Calderón-Zygmund kernel on Ω_ζ, with norm $O((s^2+\mu^2)^{-1})$, $T_s v$ is a translation of v by s.

Singular Radon transforms

$$Rv(\zeta) = \int_{\Omega_\zeta} K(\zeta,\eta) v(\eta) d\sigma_{\Omega_\zeta}(\eta)$$

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- Group Fourier transform proof by Geller-Stein
- Analogue of the Hilbert transforms along curves introduced by Nagel-Riviere-Wainger
- $WF(R) = N^*(\mathcal{C}) \cup \Delta$: works of Guillemin, and especially Greenleaf-Uhlmann on Gelfand's problem, namely to identify family of curves that suffice to invert the X-ray transform along curves.
- Most general version of L^p boundedness by Christ-Nagel-Stein-Wainger

Generalized Radon transforms

Let X, Y be smooth manifolds, and $\mathcal{C} \subset X \times Y$ a smooth submanifold. Then a Dirac measure $\delta_{\mathcal{C}}(x, y)$ supported on $\mathcal{C}$ defines a generalized Radon transform,

$$Rf(x) = \int_{C_x} \delta_C(x, y) f(y)$$

with $C_x = \{y \in Y; (x, y) \in \mathcal{C}\}$.

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The framework of Fourier integral operators

- If $\mathcal{C} = \{\varphi_1(x, y) = \cdots = \varphi_\ell(x, y) = 0\}$ locally, then

  $$\delta_{\mathcal{C}}(x, y) = \int e^{i \sum_{k=1}^{\ell} \theta_k \varphi_k(x,y)} a(x, y, \theta) d\theta,$$

  so R is a Fourier integral operator with Lagrangian $\Lambda = N^*(\mathcal{C}) \subset T^*(X) \times T^*(Y)$.

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  so R is a Fourier integral operator with Lagrangian $\Lambda = N^*(\mathcal{C}) \subset T^*(X) \times T^*(Y)$.
- General theory of Hörmander: if Λ is a local graph over $T^*(X)$ (equivalently, over $T^*(Y)$), then R is smoothing of order $(n - \ell)/2 = \dim C_x/2$.
- The local graph condition can be written down explicitly as, $\forall \theta \in \mathbf{R}^\ell \setminus 0$,

  $$\det \begin{pmatrix} 0 & d_y \varphi_j \\ d_x \varphi_k & d^2_{xy} \sum_{m=1}^{\ell} \theta_m \varphi_m(x, y) \end{pmatrix} \neq 0.$$

- In general, R is smoothing of order $\frac{1}{2}(\dim C_x - \dim \operatorname{Ker} d\pi_X)$, with π_X the projection $\pi_X : T^*(X \times Y) \to T^*(X)$.

Dirac measure of subvarieties

Consider the case $X = Y = \mathbf{R}^n$, and C_x is the translate to x of a submanifold V passing through the origin. Then the order of smoothing of R is the rate of decay of the Fourier transform of the Dirac measure on V,

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- When V is a hypersurface, the graph condition holds when the Gaussian curvature of V is not 0. The Radon transform R is then smoothing of order $\delta = (n-1)/2$.
- When V is a curve, the graph condition cannot hold if $\dim X \geq 3$. Hörmander's theorem shows only that R is smoothing of order $\delta = 0$.
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- When V is a curve with torsion, the van der Corput lemma shows that R is smoothing of order $\delta = 1/n$.
- Higher codimension lead to higher order degeneracies, which are beyond the scope of the standard method of stationary phase, and the corresponding conditions on second order derivatives.

A closer look

Let $\mathbf{R}^d \ni t \to x(t) \in \mathbf{R}^n$ be a local parametrization of V. Then

$$\hat{\delta}_V(\xi) = \int e^{i\sum_{j=1}^n \xi_j x_j(t)} \chi(t)dt$$

Setting $\xi = \lambda\omega$, $\lambda = |\xi|$, $\omega \in S^{n-1}$,

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Analytic questions

- Given $\Phi_\omega(t)$, what is the decay rate $C_\omega|\lambda|^{-\delta_\omega}$ of the above oscillatory integral with phase $\Phi_\omega(t)$?
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Geometric questions

- Given $n = \dim X$, $d = \dim V$, what are the best possible orders $\delta(n, d)$ of smoothing ? (e.g., $\delta(n, n-1) = (n-1)/2$, $\delta(n, 1) = 1/n$)
- What are the geometric conditions on V which guarantee this best possible order of smoothing ?

Multiplicities or Milnor numbers

Intuitively, smoothing should require the map $V \times \cdots \times V \to x_1 + \cdots + x_N \in \mathbf{R}^n$ to be locally surjective for N large enough. This implies that no direction $\emptyset \in \mathbf{R}^n$ is orthogonal to V at N points. This suggests a measure μ of (higher-order) curvature of V is the maximum number of points admitting a given direction among its normals. Set then, for $f \in C^\omega$, and a an isolated critical point of f,

$$\mu = \dim \mathcal{A}(a)/\mathcal{I}[\partial_1 f, \cdots, \partial_d f]$$

with $\mathcal{A}(a)$ the space of germs of analytic functions at a, and $\mathcal{I}[\partial_1 f, \cdots, \partial_d f]$ the ideal generated by the partial derivatives of f at a. We say that V has non-vanishing μ-curvature if $\forall \omega \in \mathbf{R}^n \setminus 0$, the phase $\Phi_\omega(t)$ has multiplicity at most μ at any critical point. (Note that $\mu = 1$ for a hypersurface and $\mu = n$ for a curve correspond respectively to non-vanishing Gaussian curvature and non-vanishing torsion).

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- A naive conjecture for the optimal order of smoothing for Radon transforms defined by d-dimensional submanifolds with non-vanishing curvature is

$$\delta = \frac{d}{\mu^{\frac{1}{d}} + 1} \tag{0.1}$$

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- This reduces to $(n-1)/2$ for hypersurfaces with non-vanishing curvature, and $1/n$ for curves with torsion.
- For general d, μ provides a non-linear interpolation between these extreme cases.
- The case $d = 2$ can be proved (P.-Stein, with a loss of ϵ derivatives, ϵ arbitrarily small), using results of Varchenko, Karpushkin, and Kushnirenko.

Estimates for Degenerate Oscillatory Integrals

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The van der Corput lemma

Let $\Phi(t)$ be a smooth real-valued function on $[a, b]$. If $|\Phi^{(k)}(t)| \geq 1$ then

$$|\int_a^b e^{i\lambda\Phi(t)} dt| \leq C_k\, |\lambda|^{-\frac{1}{k}}$$

if $k \geq 2$, or $k = 1$ and $\Phi'(t)$ is monotone. Here C_k is a constant depending only on k.

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Varchenko's theorem

- Let $\Phi(t)$ be a real-valued function on $\mathbf{R}^d$ with 0 as a critical point. The Newton diagram of Φ is the convex hull of the upper quadrants in $\mathbf{R}^d$ with vertices at those $k=(k_1,\cdots,k_d)$ with the monomial t^k appearing in the Taylor expansion of Φ. The Newton distance α is defined by the condition that $(\alpha^{-1},\cdots,\alpha^{-1})$ be the intersection of the line $k_1=\cdots=k_d$ with a face of the Newton diagram.
- For each face γ of the Newton diagram, let P_γ be the polynomial in the Taylor expansion of Φ with monomials in the face γ. Assume $dP_\gamma \neq 0$ in $\mathbf{R}^d \setminus 0$. Then

$$|\int e^{i\lambda\Phi(t)}\chi(t)dt| \leq C\,|\lambda|^{-\alpha}(\log|\lambda|)^{\beta}.$$

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Stability theorem of Karpushkin

Let Φ be a C^ω function with $\int_{|t|<r}|\Phi(t)|^{-\delta} < \infty$ for some $\delta > 0$, $d = 2$. Then there exists $0 < s < r$ and $\epsilon > 0$ so that for all C^ω Ψ with $\|\Phi - \Psi\|_{C^0(|t|<r, t\in\mathbf{C}^2)} < \epsilon$,

$$\int_{|t|<s}|\Psi|^{-\delta} < \infty$$

Criteria for adapted coordinate systems

- The decay rate δ is reparametrization invariant, while the Newton distance α is coordinate dependent. The equality

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- In dimension $d = 2$, roots $r_j(x)$ of polynomials $\Phi(x, y)$ are given by Puiseux series. Define the "clustering Ξ" to be the number of elements in the largest cluster of roots, where a cluster of roots is an equivalence class of roots, with $r_j \sim r_k$ if $|r_j - r_k| \cdot |r_j|^{-1} \to 0$. The criterion for adapted coordinate systems is

$$\Xi \leq \alpha^{-1}$$

and such coordinate systems always exist (P.-Stein-Sturm; also simpler and self-contained proof of the stability theorem of Karpushkin).

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can only hold generically. It is very useful to have criteria for when it holds.
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$$\Xi \leq \alpha^{-1}$$

and such coordinate systems always exist (P.-Stein-Sturm; also simpler and self-contained proof of the stability theorem of Karpushkin).
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The recent work of Collins-Greenleaf-Pramanik

- In general dimension d, for a given non-constant C^ω function Φ, there exists a finite collection $\mathcal{C}$ of coordinate transformations, so that if $\alpha(F)$ denotes the Newton distance in the coordinate system $F \in \mathcal{F}$, we have

$$\delta = \inf_{F \in \mathcal{C}} \alpha(F).$$

- The construction of the class $\mathcal{C}$ is actually algorithmic.
- Criteria for whether a specific coordinate system in $\mathcal{C}$ is adapted can be formulated in terms of projections onto diagrams in 2 variables, and using the 2-dimensional criteria formulated above.

Estimates for Sublevel Sets

It is not difficult to see that the decay rate of oscillatory integrals with phase Φ is essentially the same as the growth rate of the volume of its level sets

$$|\{t \in B; |\Phi(t)| \leq M\}| \leq C\, M^{\delta}$$

Stable estimates for oscillatory integrals correspond to volume estimates with bounds C uniform in Φ. In fact, certain even stronger bounds are known, which depend only on lower bounds for certain derivatives of Φ.

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Sublevel set estimates of Carbery-Christ-Wright

For any multi-index k, there exists $\delta > 0$ and C, depending only on k and d, so that the above estimate holds, for any function Φ satisfying the lower bound

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Sublevel set operators of P.-Stein-Sturm

For any given set $\beta^{(1)}, \cdots, \beta^{(K)} \in \mathbf{N}^d \setminus 0$, define the multilinear operator

$$W_M(f_1, \cdots, f_d) = \int_{|\partial^{\beta^{(j)}}\Phi| > 1,\ 1 \leq j \leq K} f_1(x_1) \cdots f_d(x_d)\, dx_1 \cdots dx_d$$

Assume that Φ is a polynomial of degree m. Then there exists a constant C depending only $\beta^{(1)}, \cdots, \beta^{(K)} \in \mathbf{N}^d \setminus 0$, so that

$$|W_M(f_1, \cdots, f_d)| \leq C\, M^{\frac{1}{d}\alpha} \log^{d-2}(2 + \frac{1}{M}) \prod_{i=1}^{d} \|f_i\|_{L^{\frac{d}{d-1}}}$$

where α is the Newton distance for the diagram with vertices at $\beta^{(j)}$, $1 \leq j \leq K$.

The new estimates of Gressman

- Define inductively the classes $\mathcal{L}^{\kappa,\rho}$ of operators by
  $\mathcal{L}^{1,(0,\cdots,0)}$ consists of the identity;
  $\mathcal{L}^{\kappa,\rho}$ consists of operators of the form

$$L\Phi = \det \begin{pmatrix} \partial_{t_{i_1}} L_1\Phi & \cdots & \partial_{t_{i_1}} L_n\Phi \\ \cdot & \cdot & \cdot \\ \partial_{t_{i_n}} L_1\Phi & \cdots & \partial_{t_{i_n}} L_n\Phi \end{pmatrix}$$

  Here $L_p \in \mathcal{L}^{\kappa_p,\rho_p}$, $\kappa = \kappa_1 + \cdots + \kappa_n$, $\rho = \rho_1 + \cdots + \rho_p + (1,\cdots,1,0,\cdots,0)$, the 1 occurring at i_p.
- If $\Phi \in C^\omega(B)$, and for any closed set $D \subset B$, there exists a constant C so that

$$|\{t \in D; |\Phi(t)| \leq M\}| \leq C\, M^{\frac{\alpha}{|\beta|+1-\alpha}} (\inf_{t\in D}|L\Phi|)^{-\frac{1}{|\beta|+1-\alpha}}$$

  For Pfaffian functions Φ, C depends only on d, L, and the Pfaffian type of Φ.
- The proof makes use of works of Khovanskii and Gabrielov.

Degenerate Oscillatory Integral Operators

Low order of degeneracies

- Lagrangians with two-sided Whitney folds: smoothing with loss of $\frac{1}{6}$ derivatives (Melrose-Taylor)
- Lagrangians with one-sided Whitney fold: smoothing with loss of $\frac{1}{4}$ derivatives (Greenleaf-Uhlmann)
- Lagrangians with two-sided cusps: loss of $\frac{1}{4}$ (Comech-Cuccagna, Greenleaf-Seeger)
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Arbitrary degeneracies in $1 + 1$ dimensions

- (P.-Stein) Let $\Phi(x, y)$ be a real-analytic phase function in 2 dimensions. Then the oscillatory integral operator T_λ defined by

$$Tf(x) = \int_{\mathbf{R}} e^{i\lambda\Phi(x,y)}\chi(x, y)f(y)dy$$

for $\chi \in C_0^\infty(\mathbf{R}^2)$ with sufficiently small support near 0, is bounded on $L^2(\mathbf{R})$ with norm

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- Extensions to C^∞ phases were obtained by Rychkov, Greenblatt. A simpler proof for polynomial phases was given later by P.-Stein-Sturm, using sublevel set multilinear functionals, and the Hardy-Littlewood maximal function.

- (P.-Stein) Let $\Phi(x, y)$ and $\chi(x, y)$ be as previously. Then the damped oscillatory integral operator

$$Df(x) = \int_{\mathbf{R}} e^{i\lambda\Phi(x,y)} |\Phi''_{xy}(x, y)|^{\frac{1}{2}} \chi(x, y) f(y) dy$$

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Related non-oscillating operator

(P.-Stein) Let E be the following operator, where I is a small interval around 0,

$$Ef(x) = \int_I |\Phi(x,y)|^{-\mu} f(y)\,dy$$

Then E is a bounded operator on $L^2(\mathbf{R})$ for

$$\mu < \frac{1}{2}\delta_0$$

where δ_0 is the Newton distance for Φ at 0. It is still bounded on $L^2(\mathbf{R})$ when $\mu = \frac{1}{2}\delta_0$, except possibly when the main face reduces to a single vertex, or is parallel to one of the axes, or to the line $p+q=0$.

The general strategy: "operator van der Corput"

- Decompose the set $\{\Phi''_{xy} \neq 0\}$ into $\{\Phi''_{xy} \neq 0\} = \cup_k \{|\Phi''_{xy}| \sim 2^{-k}\}$ with corresponding partition $\chi_k(x, y)$ and decomposition $T = \sum_k T_k$.

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- The sets $\{|\Phi''_{xy}| \sim 2^{-k}\}$ are usually very complicated geometrically, and the partition $\chi_k(x,y)$ necessarily complicated also. It is essential that the oscillatory estimate be uniform in χ_k. and this requires very precise versions of the oscillatory integral estimates.

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- Curved Box Lemma: Let a curved box $\mathcal{B}$ be a set of the form

$$\mathcal{B} = \{(x,y); \phi(x) < y < \phi(x) + \delta,\ a < x < b\}$$

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- Curved Trapezoid Lemma: requires Hardy-Littlewood maximal function (P.-Stein-Sturm)

Further Developments

Works of Kempe-Ikromov-Müller

L^p boundedness of maximal Radon transforms for smooth hypersurfaces in $\mathbf{R}^3$, $p > h(S)$, where $h(S)$ is the supremum over Newton distances. Applications to conjectures of Stein, Iosevich-Sawyer, and to restriction theorems.

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- Tang's result: Let $\Phi(x, z) = \sum_{j=1}^{m-1} P_j(x) z^{m-j}$ be a homogeneous polynomial of degree m in $\mathbf{R}^2 \times \mathbf{R}$. Assume that the first and the last non-vanishing polynomials $P_{j_{min}}$ and $P_{j_{max}}$ are non-degenerate ($dP(x) \neq 0$ for $x \neq 0$), and that $j_{min} \leq \frac{2m}{3} \leq j_{max}$. Then for $m \geq 4$,

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$$\|T\| \leq C\,|\lambda|^{-\frac{n_X+n_Y}{2m}} \quad \text{if} \quad m > n_X + n_Y,$$

  and $\|T\| \leq C|\lambda|^{-1/2} \log|\lambda|$ if $m = n_X + n_Y$, and $\|T\| \leq C|\lambda|^{-1/2}$ if $2 \leq m < n_X + n_Y$.

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- Cubic phases: Greenleaf-Pramanik-Tang ($n_X = n_Y = 2$); also Gressman.

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- Can this lead to a geometry on the space of phase functions, which can help identify compact sets within the subspace of phase functions with s specific volume growth rate ?